Indian Statistical Institute, Bangalore
M. Math. Second Year
First Semester - Advanced Probability

Semestral Exam — Date : Nov 10, 2014
Max Marks:100 — Time:3 hours

1. Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables.
   (a) Show that $P\{| X_n |> n \text{ infinitely often}\} = 0$ iff $X_1 \in L^1(P)$. [8]
   (b) Suppose that $\frac{X_1+...+X_n}{n} \to Y$ almost surely, for some random variable Y. Show that $X_1 \in L^1(P)$ and $Y = EX$ almost surely. [7]

2. (a) Show that the distribution of a bounded random variable is characterized by its moments. [6]
   (b) Let $Y \sim N(0,1)$. Let $X := e^Y$. Show that X has density
   $$f(x) = \frac{1}{\sqrt{2\pi}}\ \frac{1}{x}\ e^{-(\log x)^2},\ 0 < x < \infty$$
   and has moments $EX^n = e^{\frac{n^2}{2}}, n \geq 1$ [3]
   (c) Show that for $n \geq 0$,
   $$\int_0^\infty x^n f(x)\ \sin(2\pi \log x)\ dx = 0$$
   where $f(x)$ as in 2(b). Hence deduce that the result in (a) is false for unbounded random variables. [6]

3. (a) Let μ be a probability measure on $\mathbb{R}$ with $\mu(\mathbb{Z}) = 1$. Let φ be the characteristic function of μ. Show that if $x \in \mathbb{Z}$,
   $$\mu\{x\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-itx}\varphi(t)\ dt$$
   [9]
   (b) Show that for $\alpha > 2, \varphi_\alpha(t) := e^{-|t|^\alpha}$is not a characteristic function.
   Hint: If ϕ_α were the characteristic function of X, consider Var (X) [6]
   (c) If $X = (X_1,\ ...\ X_n)$ is a random vector in $\mathbb{R}^n, \lambda \in \mathbb{R}^n$ and $\lambda\ .\ x = \sum_{i=1}^{n} \lambda_i\ x_i$, show that $\sigma(X) = \sigma\{\lambda.X,\ \lambda \in \mathbb{R}^n\}$. Hence show that X is independent of the r.v Y iff for all $\lambda \in \mathbb{R}^n, \lambda\ .\ X$ is independent of Y. [5]

4. (a) Let $\{X_n, n \geq 1\}$ be an i.i.d sequence with $EX_1 = 0, EX_1^2 = 1$. Let $S_n = \frac{X_1,+\ ...\ +X_n}{\sqrt{n}}$. Let $\varphi_n(t)$ be the characteristic function of S_n. Suppose $\varphi_n(t) \to e^{-t^2/2}$ for all $t \in \mathbb{R}$. Then show (without using the Levy continuity theorem) that S_n converges weakly to the standard normal distribution. [10]
   (b) Let $\{X_{n,l}, 1 \leq l \leq k_n\}$ be a triangular array with $EX_{n,l} = 0, \sum_{l=1}^{k_n} EX_{nl}^2 = 1$ and for each $n \geq 1, \{X_{n,l},\ 1 \leq l\ \leq k_n\}$ are independent r.v. Suppose $\{X_{n,l}\}$ satisfy the Lindberg condition. Then show that $\{X_{n,l}\}$ is a null array i.e., for all $\epsilon > 0$,
   $$\lim_{n\to\infty}\ \max_{1\leq l\leq k_n} P\{| X_{n,l} |> \epsilon\} = 0.$$
   [5]

5. Let for $x \in \mathbb{R}, p(x,y)$ be a probability density on $(\mathbb{R}, \mathcal{B})$. Let $p(x,y)$ be jointly Borel measurable. Using the Kolmogorov consistency theorem construct a discrete time, continuous state space Markov Chain $\{X_n\}_{n\geq 0}$ with $X_0 \equiv 0$ and

$$P(X_n \in B \mid X_{n-1} = x) = \int_B p(x,y)dy.$$

[10]

6. Let $(X_t)_{t\geq 0}$ be a standard one dimensional Brownian motion.

   (a) Define $X_t^0 := X_t - tX_1, 0 \leq t \leq 1$. Show that (X_t^0) is a Gaussian process and compute its mean and covariance function. Note that $X_0^0 = X_1^0 \equiv 0$. [6]

   (b) Show that $\sigma\{X_t^0, 0 \leq t \leq 1\}$ is independent of $\sigma\{X_1\}$. Hint: the former σ-field is generated by finite dimensional events. [6]

   (c) Let $X_\cdot(\omega), X_\cdot^0(\omega)$ denote the trajectories $t \to X_t(\omega), t \to X_t^0(\omega)$, in $C[0,1]$. Let P^0 be the law of (X_t^0) on $C[0,1]$ i.e., for $A \subset C[0,1]$, a Borel set, $P^0(A) := P(X_\cdot^0 \in A) = P \circ (X_\cdot^0)^{-1}(A)$. Show that $P^0(A) = P(X_\cdot^0 \in A \mid X_1 = 0)$ by showing that

$$\lim_{\epsilon \downarrow 0} \frac{P(X_\cdot \in A, -\epsilon < X_1 < \epsilon)}{P(-\epsilon < X_1 < \epsilon)} = P^0(A)$$

   for every Borel set A in $C[0,1]$. [8]

7. Let $(X_t)_{t\geq 0}$ be a standard d-dimensional Brownian motion.

   (a) Let $\lambda \in \mathbb{R}^d$. Show that $\{e^{\lambda \cdot X_t - \frac{|\lambda|^2}{2}t}, t \geq 0\}$ is a martingale with respect to the natural filtration of (X_t). [5]

   (b) Let $x \in \mathbb{R}^d$. Show that almost surely, the set $\{t : X_t = x\} \subset [0,\infty)$ has Lebesgue measure zero. [5]